Category Theory | Practice Exam 21 | Name:
Dr. Paul L. Bailey | Tuesday, November 19, 2019

Problem 1 (Gallian 2.25)**.** Suppose the table below is a Cayley table for a group. Fill in the blanks.

	e	a	b	c	d
e	e	____	____	____	____
a	____	b	____	____	e
b	____	c	d	e	____
c	____	d	____	a	b
d	____	____	____	s ____	____

Problem 2 (Gallian 2.34)**.** Set

$$H = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \in \mathbf{GL}_3(\mathbb{R}) \middle| a, b, c \in \mathbb{R} \right\}.$$

Show that $H \leq \mathbf{GL}_3(\mathbb{R})$. This is called the *Heisenberg group*.

Problem 3. Let G be a group. The *center* of G is

$$Z(G) = \{z \in G \mid zg = gz \text{ for all } g \in G\}.$$

Show that $Z(G) \leq G$.

Problem 4. '[Gallian 3.20] Let G be a group. Let $H \leq G$ and $X \subset G$. The *centralizer of X in H* is

$$C_H(X) = \{h \in H \mid hxh^{-1} = x \text{ for all } x \in X\}.$$

Show that $C_H(X) \leq H$. Find an example where $X \leq G$ but $C_H(X)$ is not abelian.

Problem 5. Let G be a group. Let $H \leq G$ and $X \subset G$. The *normalizer of X in H* is

$$N_H(X) = \{h \in H \mid hXh^{-1} = X\}.$$

Show that $N_H(X) \leq X$.

Remark 1. Write $C_H(g)$ to mean $C_H(\{g\})$. Also, write $C(X)$ to mean $C_G(X)$. Similarly with normalizers.

Problem 6 (Gallian 4.16)**.** Find a collection of subngroups $\langle a_1 \rangle, \langle a_2 \rangle, ..., \langle a_n \rangle$ of $\mathbb{Z}_{240}$ with the property that $\langle a_1 \rangle \subset \langle a_2 \rangle \subset \cdots \subset \langle a_n \rangle$, with n as large a possible.

Problem 7 (Gallian 4.32)**.** Determine the subgroup lattice for $\mathbb{Z}_{12}$.

Problem 8. Determine the subgroup lattice for A_4.

Problem 9. Determine the subgroup lattice for D_6.

Problem 10. The set of subgroups of a given group form a partially ordered set, ordered by containment. Are any of $\mathbb{Z}_{12}$, A_4, or D_6 isomorphic to each other in the category of posets?

Problem 11 (Gallian 5.27)**.** Let $\beta \in S_7$ such that $\beta^4 =$ `(2 1 4 3 5 6 7)`. Find β.

Problem 12 (Gallian 5.29)**.** Find three elements σ in S_9 with the property that $\sigma^3 =$ `(1 5 7)(2 8 3)(4 6 9)`.

Problem 13 (Gallian 6.14)**.** Find Aut($\mathbb{Z}_6$ in the category of groups.

Problem 14. Find Aut($\mathbb{Z}_{10}$) in the category of group. Then, find Aut($\mathbb{Z}_{10}$) in the category of rings.

Problem 15. Let $\mathbb{Q}^+ = \{x \in \mathbb{Q} \mid x > 0\}$; this is a group under multiplication. Show that $\mathbb{Q}^+$ is isomorphic to a proper subgroup of itself.

Problem 16 (Gallian 7.27)**.** Let G be a group with subgroups H and K such that $K \subset H \subset G$. Show that

$$[G : K] = [G : H][H : K].$$

Problem 17 (Gallian 9.57)**.** Let G be a finite group and let $K \lhd G$. Show that if G/K has an element of order n, then G has an element of order n. Show by example that this is not necessarily true if G is not finite.

Problem 18 (Gallian 9.52)**.** Let G be a group and let $g, h \in G$. The *commutator* of g and h is

$$[g, h] = g^{-1}h^{-1}gh.$$

The *commutator subgroup* of G, denoted G', is the subgroup of G generated by all of the commutators:

$$G' = \langle\{\rangle[g, h] \mid g, h \in G\}.$$

This is the smallest subgroup of G which contains all the commutators, and is defined to be the intersection of all subgroups of G which contain all of the commutators.

(a) Show that $G' \lhd G$.

(b) Show that G/G' is abelian.

Problem 19. Consider a standard deck of 52 cards.

(a) A *Ukrainian shuffle* (perfect shuffle) consists of splitting the deck in half, and interleafing the cards so that the new top card is the original 27^{th} card; that is, the top card of the original bottom half becomes the new top card. Write a permutation of the 52 cards which describes this. How many Ukrainian shuffles does it take to return the deck to its original state?

(b) A *Russian shuffle* consists of splitting the deck in half, and interleafing the cards so that the new top card is the original top card. Write a permutation of the 52 cards which describes this. How many Russian shuffles does it take to return the deck to its original state?

Problem 20. Let $G = \mathbf{GL}_2(\mathbb{R})$, and let $A \in G$ be given by

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}.$$

Find $C(A)$.

Problem 21. Let p be the smallest positive prime integer, and let G be a group of order p^2. Show that G has a normal subgroup of order p.

Problem 22. A group of order 35 acts on a set of cardinality 6. Show that the action is not faithful.

Problem 23. Let $G = \mathbf{GL}_3(\mathbb{Z}_2)$ be the group of invertible 3×3 matrices with entries from $\mathbb{Z}_2$. Let $X = \mathbb{Z}_2^3$.

(a) Find $m = |G|$.

(b) Find $n = |X|$.

(c) Is it possible for a two-point stabilizer to act transitively on the remaining points?